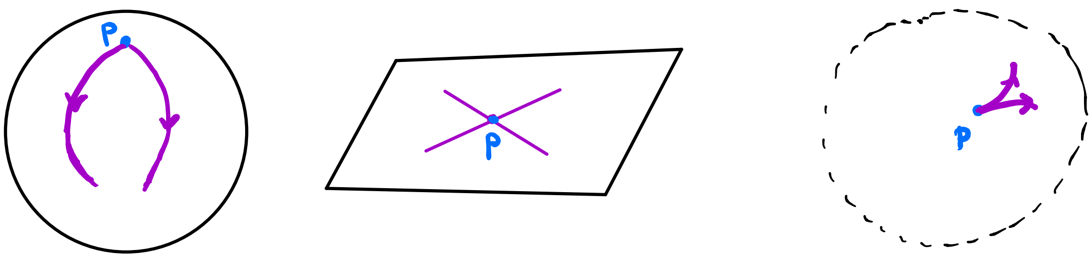
# § $1^{st}$ and $2^{nd}$ Variation formula for curves

Q: Given $(M^n, g)$, how does "curvatures" affect "geometry"?

Recall: The effect of "Gauss curvature" on geodesics in surfaces

$K>0$ $(\mathbb{S}^2)$ $\qquad$ $K=0$ $(\mathbb{R}^2)$ $\qquad$ $K<0$ $(\mathbb{H}^2)$




geodesic "converges" $\qquad$ geodesic "diverges"

Q: What about in higher dim'l?

"A": "curvatures" affect the "stability" of geodesics or more general, of minimal submanifolds.

$\rightsquigarrow$ $1^{st}$ & $2^{nd}$ variation for length/energy functional on curves!

Remark: This is like computing the gradient and hessian of a function $f: \mathbb{R}^n \to \mathbb{R}$ to determine the local min/max property at a critical point. but we are doing things in an $\infty$-dimensional setting.

# $1^{st}$ & $2^{nd}$ Variation Formula (for length / energy)

**Def$^{n}$**: Given a (piecewise) smooth curve $\gamma : [a,b] \to (M,g)$

define **Length** $L(\gamma) := \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}\, dt$ ← (indep. of reparametrization)

**Energy** $E(\gamma) := \frac{1}{2} \int_a^b g(\gamma'(t), \gamma'(t))\, dt$ ← (depending on parametrize)

**Remark**: By Hölder inequality

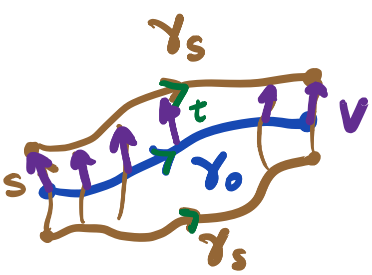
$$L(\gamma) \le \sqrt{2}\sqrt{b-a}\; E(\gamma)^{\frac{1}{2}}$$

and "=" holds $\iff$ $\|\gamma'(t)\| := \sqrt{g(\gamma'(t), \gamma'(t))} \equiv \text{const}$

$\Rightarrow$ $L$ & $E$ are the same (up to a multiplicative constant) for curves parametrized proportional to arc length.

**Setup**: Consider a 1-parameter family of smooth curves in $(M,g)$

$$\gamma(t,s) := \gamma_s(t) : \overbrace{[a,b]}^{t} \times \overbrace{(-\varepsilon, \varepsilon)}^{s} \longrightarrow M \qquad \text{smooth}$$




Look at the function of $s$

$$L(s) := L(\gamma_s)$$

$$E(s) := E(\gamma_s)$$

**Goal**: Compute $L'(0)$, $L''(0)$ and $E'(0)$, $E''(0)$

**Notation**: write the variation vector field as

$$V(t) := \frac{\partial \gamma}{\partial s}(t, 0) \qquad \text{a vector field along } \gamma_0$$

We start with the $1^{st}$ variation.

$1^{st}$ variation formula:

$$E'(s) = -\int_a^b \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle dt + \left\langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle \Bigg|_{t=a}^{t=b}$$

and $$L'(s) = \int_a^b \frac{1}{\left\| \frac{\partial \gamma}{\partial t} \right\|} \left( \frac{\partial}{\partial t} \left\langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle - \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle \right) dt$$

Proof: $E(s) := E(\gamma_s) := \frac{1}{2} \int_a^b \left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right\rangle dt$

$$\Rightarrow E'(s) = \frac{d}{ds} \left( \frac{1}{2} \int_a^b \left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right\rangle dt \right)$$

$$= \frac{1}{2} \int_a^b \frac{\partial}{\partial s} \left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right\rangle dt$$

metric compatible $\rightarrow$ $$= \int_a^b \left\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right\rangle dt$$

torsion-free

Recall: $\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right] = 0$ $\rightarrow$ $$= \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle dt$$

metric compatible $\rightarrow$ $$= \int_a^b \left( \frac{\partial}{\partial t} \left\langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle - \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle \right) dt$$

$$= -\int_a^b \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle dt + \left\langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle \Bigg|_{t=a}^{t=b}$$

Similarly, $L(s) := \int_a^b \sqrt{\left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right\rangle}\, dt$

$$\Rightarrow L'(s) = \int_a^b \frac{1}{\left\| \frac{\partial \gamma}{\partial t} \right\|} \left\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right\rangle dt$$

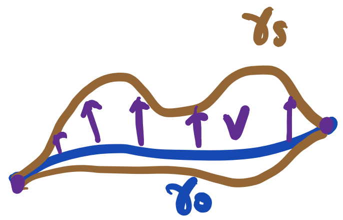
$$= \int_a^b \frac{1}{\left\| \frac{\partial \gamma}{\partial t} \right\|} \left( \frac{\partial}{\partial t} \left\langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle - \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle \right) dt$$

$\blacksquare$

If the end points are fixed in the variation :

ie $\quad \gamma_s(a) = \gamma_0(a) \;\&\; \gamma_s(b) = \gamma_0(b) \quad \forall s \in (-\varepsilon, \varepsilon)$

then $V(a) = 0 = V(b)$ and




$$E'(0) = -\int_a^b \langle V(t), \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma_0}{\partial t} \rangle \, dt$$

$$L'(0) = -\frac{1}{\|\frac{\partial \gamma_0}{\partial t}\|} \int_a^b \langle V(t), \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma_0}{\partial t} \rangle \, dt$$

provided that $\|\frac{\partial \gamma_0}{\partial t}\| \equiv$ const.

Cor: crit. pts of $E \iff \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma_0}{\partial t} \equiv 0$, ie. $\gamma_0$ is geodesic

crit. pts of $L \iff \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma_0}{\partial t} \equiv 0$, ie. $\gamma_0$ is geodesic

provided that $\|\frac{\partial \gamma_0}{\partial t}\| \equiv$ const.

Next, we assume $\gamma_0$ is a geodesic, ie $\nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma_0}{\partial t} \equiv 0$

and compute the $2^{nd}$ variation, with end pts fixed.

2nd variation formula :

$$E''(0) = \int_a^b \left( \langle \nabla_{\frac{\partial}{\partial t}} V, \nabla_{\frac{\partial}{\partial t}} V \rangle - \langle R\left(\frac{\partial \gamma_0}{\partial t}, V\right) \frac{\partial \gamma_0}{\partial t}, V \rangle \right) dt$$

$$L''(0) = \frac{1}{\|\frac{\partial \gamma_0}{\partial t}\|} \int_a^b \left( \langle \nabla_{\frac{\partial}{\partial t}} V^N, \nabla_{\frac{\partial}{\partial t}} V^N \rangle - \langle R\left(\frac{\partial \gamma_0}{\partial t}, V^N\right) \frac{\partial \gamma_0}{\partial t}, V^N \rangle \right) dt$$

Let us give a summary again:

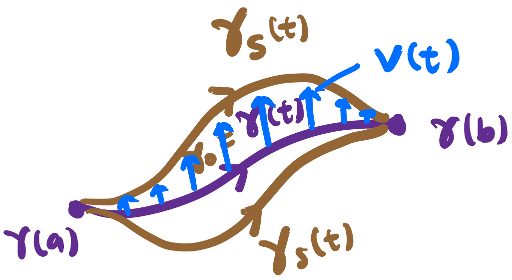
## Setting:

Given a curve $\gamma(t) : [a,b] \to (M^n, g)$, defined

$$E(\gamma) := \frac{1}{2}\int_a^b g(\gamma', \gamma')\, dt \qquad \text{energy} \leftarrow \text{dep. on parametrization}$$

$$L(\gamma) := \int_a^b \sqrt{g(\gamma', \gamma')}\, dt \qquad \text{length} \leftarrow \text{indep. of parametrization}$$

Consider a 1-parameter variation of curves (with fixed end points).




$\gamma(t,s) = \gamma_s(t) : [a,b] \times (-\varepsilon, \varepsilon) \longrightarrow M$ smooth

$\rightsquigarrow$ $E(\gamma_s)$, $L(\gamma_s)$ are smooth fcn of $s$

compute $\left.\frac{d}{ds}\right|_{s=0} E(\gamma_s)$ and $\left.\frac{d^2}{ds^2}\right|_{s=0} E(\gamma_s)$

variation field

$$V(t) := \left.\frac{\partial \gamma}{\partial s}\right|_{s=0}$$

**Prop:** (1st variation formula for energy)

$$E'(s) = -\int_a^b \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle dt$$

At a critical pt $\gamma_0$ for $E$, i.e. a geodesic, then we compute

**Prop:** (2nd variation formula for energy)

$$E''(0) = \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} V, \nabla_{\frac{\partial}{\partial t}} V \right\rangle - \left\langle R\left(\frac{\partial \gamma}{\partial t}, V\right)\frac{\partial \gamma}{\partial t}, V \right\rangle dt$$

**Proof:** Recall:
$$E'(s) = -\int_a^b \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle dt$$

Differentiate w.r.t. $s$, evaluate at $s=0$.

$$E''(0) = \frac{d}{ds}\Big|_{s=0}\left( -\int_a^b \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle dt \right)$$

$$= \frac{d}{ds}\Big|_{s=0}\left( \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle dt \right)$$

metric compatible

$$= \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s}, \boxed{\nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial t}} \right\rangle + \left\langle \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle dt$$

|| torsion-free

swap

$$= \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s}, \boxed{\nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s}} \right\rangle dt$$

$$+ \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle + \left\langle R\left(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s}\right) \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle dt$$

at $s=0$

$$= \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}} V, \nabla_{\frac{\partial}{\partial t}} V \right\rangle - \left\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial s}, \underbrace{\cancel{\nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t}}}_{=0} \right\rangle$$

$=0$ $\because \gamma_0$ is geodesic

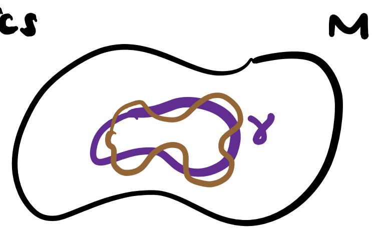
$$- \left\langle R\left(\frac{\partial \gamma}{\partial t}, V\right) \frac{\partial \gamma}{\partial t}, V \right\rangle dt$$

$\square$

Remark: One can also consider closed geodesics

$$\gamma : S^1 = \mathbb{R}/2\pi\mathbb{Z} \to M$$

without end points.




The $2^{nd}$ variation formula has important geometric and topological implications.

Note:

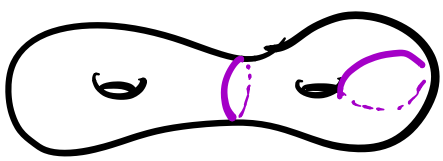
$$E''(0) = \int_a^b \underbrace{\left\| \nabla_{\frac{\partial}{\partial t}} V \right\|^2}_{\geq 0} - \underbrace{\left\langle R\left(\frac{\partial \gamma}{\partial t}, V\right) \frac{\partial \gamma}{\partial t}, V \right\rangle}_{\text{sectional curvature term}} dt$$

**Cor:** Suppose $(M^n, g)$ has negative sectional curvature, i.e. $K<0$.

THEN, any geodesic $\gamma : [a,b] \to M$ is (strictly) locally energy / length minimizing (with end points fixed).

i.e. any critical pt of $E$ or $L$ must be local minimum.

**E.g.)** On a hyperbolic surface $(\Sigma^2_{g\geq 2}, g_{hyp})$ of $K \equiv -1$.



For positively curved space, we have the following:

**Synge Theorem:** Suppose $(M^n, g)$ is a compact, oriented Riem. manifold s.t. (i) $n$ is even

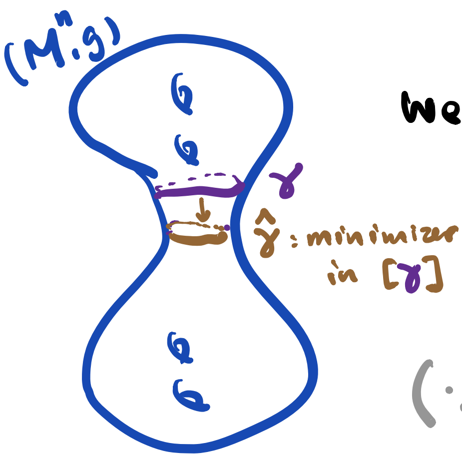
(ii) $\boxed{K>0}$ everywhere

THEN, $\pi_1(M) = 0$, i.e. $M$ is simply-connected.

**Proof:** Suppose NOT, i.e. $\pi_1(M) \neq 0$.

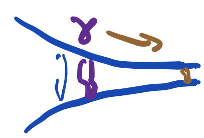
So, $\exists$ a (smooth) closed loop $\gamma : S^1 \to M$ which is NOT contractible to a pt. inside $M$, i.e. $0 \neq [\gamma] \in \pi_1(M)$

↳ free homotopy class




We want to do a minimization (w.r.t $E$) with the free homotopy class $[\gamma] \neq 0$

$$\min \{ E(\tilde{\gamma}) : \tilde{\gamma} \in [\gamma] \} = E(\hat{\gamma})$$



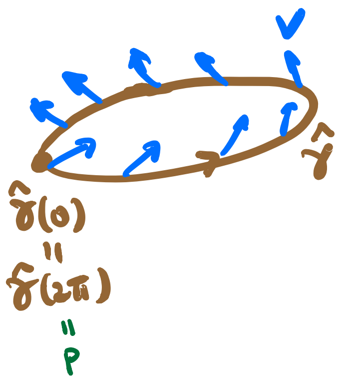

$(\because M$ cpt $\Rightarrow$ existence of minimizer $\hat{\gamma})$

Note that $\hat{\gamma}$ is a non-trivial loop since $[\hat{\gamma}] = [\gamma] \neq 0$.

AND: $E'(0) = 0$ & $E''(0) \geq 0$ at $\hat{\gamma}$

for any variation field $V$ along $\hat{\gamma}$

Write: $\hat{\gamma} : [0, 2\pi]/_{0 \sim 2\pi} \to M$ geodesic




GOAL: Find a $V$ st $E''(0) < 0$

Let $P : T_pM \xrightarrow{\text{linear}} T_pM$ be the parallel transport map along $\hat{\gamma}$ from $\underbrace{\hat{\gamma}(0)}_{=p}$ to $\underbrace{\hat{\gamma}(2\pi)}_{=p}$

$\hat{\gamma}$ geodesic $\Leftrightarrow$ $\hat{\gamma}'$ is parallel along $\hat{\gamma}$

$$\Rightarrow P(\hat{\gamma}'(0)) = \hat{\gamma}'(0)$$

Since $P$ preserve the inner product, we have

$$P : (\hat{\gamma}'(0))^\perp \underset{\cong}{\xrightarrow{\text{linear}}} (\hat{\gamma}'(0))^\perp \quad \text{ie } P \in SO(n-1)$$

dim M even $\Rightarrow$ $(\hat{\gamma}'(0))^\perp$ is odd dimensional

$$\Rightarrow \exists^{\circ} w \in (\hat{\gamma}'(0))^\perp \text{ st } P(w) = w$$

Let $V(t)$ be the unique parallel v.f. along $\hat{\gamma}$

s.t. $V(0) = V(2\pi) = w$ $\quad \nabla_{\frac{\partial}{\partial t}} V \equiv 0$

For this $V$,

$$0 > E''(0) = \int_0^{2\pi} \| \cancel{\nabla_{\frac{\partial}{\partial t}} V} \|^2 - \underbrace{\langle R(\tfrac{\partial \hat{\gamma}}{\partial t}, V)\tfrac{\partial \hat{\gamma}}{\partial t}, V \rangle}_{-\underbrace{K(\mathrm{span}\{V, \frac{\partial \hat{\gamma}}{\partial t}\})}_{<0}} \, dt$$

$\nabla_{\frac{\partial}{\partial t}} V \to 0$ $\because V$ parallel

Contradiction!

$\square$